Bifurcation Curves of Limit Cycles in some Liénard Systems

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Abstract

Liénard systems of the form $\ddot{x} + \epsilon f(x)\dot{x} + x = 0$, with $f(x)$ an even continous function, are considered. The bifurcation curves of limit cycles are calculated exactly in the weak ($\epsilon \to 0$) and in the strongly ($\epsilon \to \infty$) nonlinear regime in some examples. The number of limit cycles does not increase when ϵ increases from zero to infinity in all the cases analyzed.

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1 Introduction

Self-sustained oscillations are found very often in nature. There are many examples in different branches of science such as in biology, chemistry, mechanics, electronics, fluid dynamics, etc. [Andronov et al., 1989, Bai-Lin, 1990]. Nonlinearities are required in order to have this kind of behaviour. The system reaches an oscillatory dynamics caracterized by a preferred period, wave form and amplitude, stable under slight perturbations. The oscillations are generated by an internal balance of amplification and dissipation, even in the absence of external periodic forcing. (For instance, a nonlinear damping force which increases the amplitude for small velocities and decreases it for large velocities). This dynamical state can be modelled by the stable limit cycles found in specific nonlinear differential equations.

Limit cycles are isolated closed trajectories in phase space. They are stable if the neighbouring solutions tend to them in an asymptotic sense or unstable if the neighbouring solutions unwind from them. Determination of the number, amplitude and loci of limit cycles in a general nonlinear system is an unsolved problem that has attracted much attention in this century [Yan-Quian et al., 1986]. This constitues a part of the Hilbert's Sixteenth Problem [Hilbert, 1902] when we are restricted to two-dimensional autonomous systems of the form:

$$\dot{x} = P_n(x, y),$$

$$\dot{y} \;=\; Q_n(x,y), \tag{1}$$

where P_n and Q_n are polynomials of degree n with real coefficients. Althoug it has been proved that the number of limit cycles in systems (1) is finite [Ecalle et al., 1987, Ilyashenko, 1990], the determination of the maximal number H_n of limit cycles is still far away of being known.

The van der Pol oscillator $\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0$, where $\dot{x}(t) = dx(t)/dt$, is an example of system (1) that has been exhaustively studied. In this case, $P_3(x,y) = y$ and $Q_3(x,y) = -\epsilon(x^2-1)y - x$. It displays a limit cycle whose uniqueness and non-algebraicity has been shown for the whole range of the parameter ϵ. Its behaviour runs from near-harmonic oscillations for ϵ close to zero ($\epsilon \to 0$) to relaxation oscillations when ϵ tends to infinity ($\epsilon \to \infty$), making it a good model for many practical situations [van der Pol, 1927, López-Ruiz & Pomeau, 1997].

A generalization of the van der Pol oscillator is the Liénard equation,

$$\ddot{x} + \epsilon f(x)\dot{x} + x = 0, \tag{2}$$

with ϵ a real parameter and $f(x)$ any real function. When $f(x)$ is a polynomial of degree $N = 2n+1$ or $2n$ this equation of the form (1) with $P_{N+1}(x,y) = y$ and $Q_{N+1}(x,y) = -\epsilon f(x)y - x$. It has been conjectured by Lins, Melo and Pugh (LMP-conjecture) that the maximum number of limit cycles allowed is just n [Lins et al., 1977]. It is true if $N = 2$, or $N = 3$ or if $f(x)$ is even and $N = 4$ [Lins et al., 1977, Rychkov, 1975]. Also, it is true in the strongly nonlinear regime ($\epsilon \to \infty$) when $f(x)$

is an even polynomial [López & López-Ruiz, 2000]. There are no general results about the limit cycles when $f(x)$ is a polynomial of degree greater than 5 neither, in general, when $f(x)$ is an arbitrary real function [Giacomini & Neukirch, 1997, Giacomini & Neukirch, 1998].

In the present paper, we are interested in the Liénard equation when $f(x)$ is a continous even function, otherwise arbitrary. We exploit the fact that the calculation of the number of limit cycles in the weak ($\epsilon \to 0$) and in the strongly ($\epsilon \to \infty$) nonlinear regimes is possible for this kind of functions by means of simple algorithms. In fact, we find exactly the bifurcation curves of limit cycles in both regimes for several examples of viscous terms $f(x)$. Section 2 is devoted to explain the strategies (or algorithms) used to calculate the amplitude and number of limit cycles in those extreme regimes, and in Section 3 we analyze some particular cases found in the literature. Last Section contains the conclusions.

2 Limit Cycles in the Liénard Equation

In order to study the limit cycles of equation (2) it is convenient to rewrite it in the coordinates $(x, \dot{x}) = (x, y)$ in the plane. We perform the change of variables $\dot{x}(t) = y(x)$ and $\ddot{x}(t) = y(x)y'(x)$ (where $y'(x) = dy/dx$):

$$yy' + \epsilon f(x)y + x = 0. \tag{3}$$

A limit cycle $C_l \equiv (x, y_\pm(x))$ of equation (3) has a positive branch $y_+(x) > 0$ and a negative branch $y_-(x) < 0$. They cut the x-axis in

two points $(-a_1, 0)$ and $(a_2, 0)$ with $a_1, a_2 > 0$. The oscillation x runs in the interval $-a_1 < x < a_2$.

The origin $(0,0)$ is the only fixed point of equation (3). Then every limit cycle C_l solution of Eq. (3) encloses the origin. The result is a nested set of closed curves that defines the qualitative distribution of the integral curves in the plane (x, y). The stability of the limit cycles is alternated. For a given stable limit cycle, the two neighbouring limit cycles, the closest one in its interior and the closest one in its exterior, are unstable, and viceversa (Fig. 1).

When $f(x)$ is an even function, the symmetries of the equation (3) impose some properties over the shape of the limit cycles. Thus the *inversion symmetry* $(x, y) \leftrightarrow (-x, -y)$ implies $y_+(x) = -y_-(-x)$ and $a_1 = a_2 = a$. Therefore, we can restrict ourselves to the positive branches of the limit cycles $(x, y_+(x))$ with $-a \leq x \leq a$. The amplitude of oscillation a identifies the limit cycle. The *parameter inversion symmetry* $(\epsilon, x, y) \leftrightarrow (-\epsilon, x, -y)$ implies that if $C_l \equiv (x, y_\pm(x))$ is a limit cycle for a given ϵ, then $\overline{C}_l \equiv (x, -y_\mp(x))$ is a limit cycle for $-\epsilon$. Moreover if C_l is stable (or unstable) then $\overline{C}_l$ is unstable (or stable, respectively). Therefore it is enough to consider the positive y-branch $y_+(x)$ of the limit cycles when $\epsilon > 0$ for obtaining all the periodic solutions. (The limit cycles for a given $-\epsilon < 0$ are obtained from a reflection over the x-axis of those limit cycles obtained for $\epsilon > 0$).

Another property of a limit cycle can be derived from the fact that

the mechanical energy $E = (x^2 + y^2)/2$ is conserved in a half oscillation:

$$\int_{-a}^{a} \frac{dE}{dx} dx = 0.$$

Thus, if equation (3) is integrated along the positive branch $y_+(x)$ of a limit cycle, between the maximal amplitudes of oscillation, we obtain:

$$\int_{-a}^{a} f(x) y_+(x) dx = 0. \tag{4}$$

The solutions $y_+(x)$ of equation (3) and (4), vanishing in the extremes, constitute the finite set of limit cycles of equation (3).

2.1 The Weakly Nonlinear Regime

Liénard system (3) reduces to the simple harmonic oscillator when $\epsilon = 0$. All the circles $y(x) = \sqrt{r^2 - x^2}$ of radius r about the origin are solutions. This path-diagram is destroyed when ϵ is slightly perturbed. Only the limit cycles survive as closed curves. They will have a slightly modified circular form. At order zero in ϵ, we can suppose them as circles $y_+(x) = \sqrt{a^2 - x^2}$ with different amplitudes a's. Obviously, at this order, the condition (3) is verified, and condition (4) reads:

$$\beta(a) \equiv \int_{-a}^{a} f(x) \sqrt{a^2 - x^2} dx = 0. \tag{5}$$

Each solution $\pm a$ of the equation $\beta(a) = 0$ is the amplitude of a limit cycle of the Liénard system in the weak nonlinear regime. And viceversa, the amplitudes of all limit cycles of equation (3) are solutions of equation (5) in that regime. These results are exact for $\epsilon = 0$. In conclusion,

equation (5) determines the amplitudes of the limit cycles of Liénard system defined by $f(x)$ when $\epsilon \to 0$.

The stability of a limit cycle in this regime is given, at the lowest order in ϵ, by the sign of the integral:

$$\sigma \equiv -\int_{-a_0}^{a_0} \frac{\epsilon f(x)}{y_+(x)} = -\frac{\epsilon}{a_0}\left[\frac{d\beta(a)}{da}\right]_{a_0},$$

where $a_0 > 0$ is a solution of equation (5). The limit cycle is stable for $\sigma < 0$ and unstable for $\sigma > 0$.

As an example, we integrate equation (5) when $f(x)$ is an even polynomial of degree $2n$:

$$f(x) = b_0 + b_2 x^2 + b_4 x^4 + \cdots + b_{2n} x^{2n},$$

where $b_0, b_2, b_4, \ldots, b_{2n}$ are real coefficients. Then, only the amplitudes a that satisfy the equation:

$$\beta(a) = \frac{\pi a^2}{2} \sum_{k=0}^{n} b_{2k} \frac{(2k)!}{4^k (k+1)!\, k!} a^{2k} = 0$$

are allowed. The solution $a = 0$ corresponds to the fixed point $(0,0)$ and the factor a^2 can be eliminated. Thus, the possible amplitudes a are the zeros of an even polynomial of degree $2n$. There are no more than n different solutions $a > 0$ and therefore, the maximun number of limit cycles in this case is n. We conclude that LMP-conjecture is true in the weak nonlinear regime. For instance, $f(x) = x^2 - 1$ in the van der Pol oscillator. Then $\beta(a) = \pi a^2(a^2 - 4)/8$ and the only existing limit cycle has the amplitude $a \simeq 2$ when $\epsilon \to 0$. It is stable if $\epsilon > 0$ and unstable if $\epsilon < 0$.

2.2 The Strongly Nonlinear Regime

An algorithm that determines the number and amplitude of the limit cycles of Liénard systems in the strongly nonlinear regime has been proposed in reference [López & López-Ruiz, 2000]. A first approach to the shape of limit cycles when $\epsilon \to +\infty$ shows that the positive y-branch, $y_+(x) \equiv \epsilon z_i^s(x)$, of a stable limit cycle with amplitude $a_i^s > 0$ is given by:

$$z_i^s(x) = \begin{cases} 0 & \text{if } -a_i^s \leq x \leq s_i \\ -F(x) + F(s_i) & \text{if } \quad s_i \leq x \leq a_i^s, \end{cases} \tag{6}$$

where $F(x) = \int_0^x f(t)dt$ and $s_i < 0$ is called the *gluing point* of the two pieces $z(x) = 0$ and $z(x) = -F(x) + F(s_i)$. The unstable ones, $y_+(x) \equiv \epsilon z_i^u(x)$, with amplitude $a_i^u > 0$ are given by:

$$z_i^u(x) = \begin{cases} -F(x) + F(u_i) & \text{if } -a_i^u \leq x \leq u_i \\ 0 & \text{if } \quad u_i \leq x \leq a_i^u, \end{cases} \tag{7}$$

where $u_i > 0$ is the gluing point of the two pieces in this case.

In the remaining of this section we give a brief skecht of the algorithm (for a detailed discussion see reference [López & López-Ruiz, 2000]).

<u>STABLE CYCLES</u>: Consider the points $s^* < 0$ where $F(x)$ has a positive local maximum and find the points a^* defined by the rule:

$$a^* = \min \{x > s^*, F(x) = F(s^*)\} .$$

Geometrically a^* represents the x-coordinate of the first crossing point between the straight $z = F(s^*)$ and the curve $z = F(x)$ in the plane (x, z). If $a^* < |s^*|$ it is not possible to build the limit cycle and we can

eliminate this s^* as a possible gluing point. If $a^* > |s^*|$ the point s^* is a gluing point candidate. We rename and order all the pairs (s^*, a^*) verifying this last property as $(\bar{s}_i, \bar{a}_i^s)$ with $\bar{s}_{i+1} < \bar{s}_i < 0$ and collect them into the set:

$$\bar{\mathcal{A}}^s \equiv \{(\bar{s}_i, \bar{a}_i^s), F(\bar{s}_i) \ \ local \ \ maximum, F(\bar{s}_i) > 0, \bar{a}_i^s > |\bar{s}_i|\}\,.$$

By construction $\bar{a}_{i+1}^s > \bar{a}_i^s$. There are two different situations when two sucessive pairs, $(\bar{s}_i, \bar{a}_i^s)$ and $(\bar{s}_{i+1}, \bar{a}_{i+1}^s)$, are considered:

(a) $-\bar{a}_{i+1}^s < \bar{s}_{i+1} < -\bar{a}_i^s < \bar{s}_i$. In this case it is possible to build a two-piecewise limit cycle with the pair $(\bar{s}_i, \bar{a}_i^s)$ as indicated in Eq. (6). This pair is picked out and renamed again as (s_i, a_i^s).

(b) $-\bar{a}_{i+1}^s < -\bar{a}_i^s < \bar{s}_{i+1} < \bar{s}_i$. Now the constuction of a limit cycle derived from the pair $(\bar{s}_i, \bar{a}_i^s)$ is not possible. This pair is rejected.

If there is only one pair $(\bar{s}_1, \bar{a}_1^s)$, we consider it satisfies (a).

All the existing stable limit cycles can be found comparing the pairs i and $i+1$ under rules (a)-(b) and iterating this process. All the pairs selected by condition (a) (and renamed as (s_i, a_i^s)) are collected into the set:

$$\mathcal{A}^s \equiv \{(s_i, a_i^s)\} = \left\{(\bar{s}_i, \bar{a}_i^s) \in \bar{\mathcal{A}}^s, (\bar{s}_i, \bar{a}_i^s) \ \text{ verifies (a)}\right\}. \qquad (8)$$

The number, $l_s = card(\mathcal{A}^s)$, of pairs (s_i, a_i^s) is the number of stable limit cycles of the system (3).

UNSTABLE CYCLES: The same process can be repeated for the unstable cycles by considering the points $u^* > 0$, where $F(u^*)$ is a positive

local maximum, and their partners a^* are defined by:

$$a^* = \max\left\{x < u^*, F(x) = F(u^*)\right\}.$$

The gluing point candidates u^* must verify $|a^*| > u^*$. After renaming and ordering the pairs (u^*, a^*) fulfilling this last condition as $(\bar{u}_i, \bar{a}_i^u)$ with $\bar{u}_{i+1} > \bar{u}_i > 0$, we collect them into the set:

$$\bar{\mathcal{A}}^u \equiv \left\{(\bar{u}_i, \bar{a}_i^u), F(\bar{u}_i) \;\; local \;\; maximum, F(\bar{u}_i) > 0, |\bar{a}_i^u| > \bar{u}_i\right\}.$$

A similar algorithm as indicated above can be applied in this case with the following modified rules:

(a') $\bar{u}_i < -\bar{a}_i^u < \bar{u}_{i+1} < -\bar{a}_{i+1}^u$. In this case there exists an unstable two-piecewise limit cycle resulting from the pair $(\bar{u}_i, \bar{a}_i^u)$ and given in Eq. (7). This pair is picked out and renamed (u_i, a_i^u).

(b') $\bar{u}_i < \bar{u}_{i+1} < -\bar{a}_i^u < -\bar{a}_{i+1}^u$. The pair $(\bar{u}_i, \bar{a}_i^u)$ does not produce a limit cycle and is rejected.

If there is only one pair $(\bar{u}_1, \bar{a}_1^u)$ we consider it satisfies (a').

We iterate the process given by rules (a')-(b'). All the pairs selected by condition (a') are collected into the set:

$$\mathcal{A}^u \equiv \{(u_i, a_i^u)\} = \Big\{(\bar{u}_i, \bar{a}_i^u) \in \bar{\mathcal{A}}^u, (\bar{u}_i, \bar{a}_i^u) \;\text{ verifies (a')}\Big\}. \qquad (9)$$

The number, $l_u = card(\mathcal{A}^u)$, of pairs (u_i, a_i^u) is the number of unstable limit cycles of system (3). Obviously, $l_s - 1 \leq l_u \leq l_s + 1$.

It was claimed in [López & López-Ruiz, 2000] that the total number l of limit cycles of Eq. (3) in the strongly nonlinear regime is $l = l_s + l_u$,

where l_s and l_u are the number of stable and unstable limit cycles respectively. The amplitudes of these limit cycles are given by the numbers a_i^s and a_i^u, respectively.

We remark also that each pair of zeros $\pm x_i$ of $f(x)$ produces at most one limit cycle. If $f(x)$ is an even polynomial of degree $2n$ there will be at most n limit cycles. Therefore, LMP-conjecture is also true in the strongly nonlinear regime. For instance, in the van der Pol oscillator, $F(x) = -x + x^3/3$ has an unique positive local maximum at $s = -1$. The amplitude a of the only existing limit cycle when $\epsilon \to \infty$ is given by the solution of the relation $F(-1) = F(a)$, that is, $a = 2$. Its shape, $y_+(x) \equiv \epsilon z(x)$, is (up to order ϵ^{-2}) given by:

$$z(x) = \begin{cases} 0 & \text{if } -2 \leq x \leq -1 \\ \frac{1}{3}(-x^3 + 3x + 2) & \text{if } -1 \leq x \leq 2, \end{cases}$$

3 Bifurcation Curves in some Examples

We apply in this section the results of the former section to particular examples that have been studied by different authors in the literature.

Example 1: $\underline{\mathbf{f(x) = x^{2n} - 1}}$, with $n = 1, 2, 3, \cdots$. This case represents a generalization of the van der Pol oscillator [van der Pol, 1927]. It has only a limit cycle for the whole range of the parameter ϵ.

(i) $\underline{\epsilon \to 0}$: The amplitude a_n, $n = 1, 2, 3, \cdots$, of this limit cycle in the

weakly nonlinear regime is the solution of the equation $\beta(a_n) = 0$. We obtain:

$$a_n = 2 \sqrt[2n]{\frac{n!(n+1)!}{(2n)!}}$$

If $n = 1$ then $a_1 = 2$ (van der Pol system) and if $n \to \infty$ the result is a_∞=1. That is, $1 \leq a_n \leq 2$ for all values of n.

(ii) $\underline{\epsilon \to \infty}$: The calculation of the amplitude a_n of this limit cycle in the strongly nonlinear regime requires to find the positive local maxima of $F(x) = \int_0^x f(t)dt$. In this case, the only local maximum is at $x = -1$ with the value $F(-1) = 2n/(2n+1)$. The amplitude a_n is therefore given by:

$$a_n = F^{-1}\left(\frac{2n}{2n+1}\right) > 0$$

In particular, $a_1 = 2$ and if $n \to \infty$ the amplitude is $a_\infty = 1$. Also, in this case, $1 \leq a_n \leq 2$ for each value of n.

(iii) $\underline{0 \ll \epsilon \ll \infty}$: Computer simulations show that the amplitude a_n of the unique limit cycle is slighty perturbed in this regime.

Example 2: $\underline{\mathbf{f(x) = (x^2 - 1)(x^2 - k)}}$, where k is a real parameter (Fig. 2). This sytem was studied by Lloyd in Ref. [Lloyd, 1987]. He showed that it has no periodic solutions if $1/5 < k < 5$, and he suggested that there exist k_* and k^*, depending on ϵ, such that there are two periodic solutions if $0 < k < k_*$ or $k > k^*$, while there are none if $k_* < k < k^*$. Moreover, he finds that $k^* \leq (7 + \sqrt{45})/2$ for some positive ϵ. Here, the values of (k_*, k^*) in the weakly, (k_0, k^0), and in the strongly, (k_∞, k^∞), nonlinear regimes are calculated. For intermediate values of ϵ it is found numerically that $k_\infty \leq k_* \leq k_0$ and $k^0 \leq k^* \leq k^\infty$.

(i) $\underline{\epsilon \longrightarrow 0}$: The amplitudes $a_\pm$, solutions of the equation $\beta(a) = 0$, of the limit cycles in this regime are the positive values of the expression:

$$a_\pm = \left\{(k+1) \pm \sqrt{(k+1)^2 - 8k}\right\}^{\frac{1}{2}}$$

If $k < 0$ there is only one limit cycle of amplitude a_+. If $k > 0$ the sign of the discriminant $\Delta = (k+1)^2 - 8k$ determines the number of periodic solutions. The roots of Δ are: $k_0 = 3 - \sqrt{8} \simeq 0.17157$ and $k^0 = 3 + \sqrt{8} \simeq 5.82842$. Then, if $k_0 < k < k^0$, Δ is negative and there is no limit cycle. If $0 < k < k_0$ or $k > k^0$, Δ is positive and the system has two periodic solutions of amplitudes $a_\pm$. In $k = 0$ a limit cycle of small amplitude bifurcates from the origin after an Andronov-Hopf bifurcation. In $k = k_0$ and $k = k^0$ the two limit cycles appear or disappear by a saddle-node bifurcation.

(ii) $\underline{\epsilon \longrightarrow \infty}$: The limit cycles in this regime are determined by the the positive local maxima of $F(x) = x^5/5 - (k+1)x^3/3 + kx$. If $k < 0$ there is only a positive local maximum at $x_+ = -1$ with the value $F(-1) = (2 - 10k)/15$. The amplitude a_+ of this limit cycle is:

$$a_+ = F^{-1}\left(\frac{2-10k}{15}\right) > 0$$

If $0 < k < 1/5$ there are two positive local maxima: one is at $x_+ = -1$ and the other one is at $x_- = \sqrt{k}$. The condition for the existence of two limit cycles is: $F(-1) > F(\sqrt{k})$. This is verified when $0 < k < k_\infty = (3 - \sqrt{5})^2/4 \simeq 0.14589$. The amplitudes $a_\pm$ of these limit cycles are:

$$\begin{aligned} a_+ &= F^{-1}\left(\frac{2-10k}{15}\right) > 0 \\ a_- &= \left| max\left\{F^{-1}\left(\frac{(10-2k)k^{\frac{3}{2}}}{15}\right) < 0\right\}\right| \end{aligned}$$

If $1/5 < k < 1$ there is only a positive local maximum at $x = \sqrt{k}$ for which $F^{-1}(x)$ has not an antiimage. Then there is not periodic solutions in that interval of k. The same behaviour is found when $1 < k < 5$, but now the positive local maximum is located at $x = 1$.

If $k > 5$ there are two positive local maxima localized at $x_+ = -\sqrt{k}$ and at $x_- = 1$. The condition for having periodic solutions reads: $F(-\sqrt{k}) > F(1)$. This condition holds when $k > k^\infty = (3+\sqrt{5})^2/4 \simeq 6.85410$. The amplitudes $a_\pm$ of these limit cycles are:

$$\begin{aligned} a_+ &= F^{-1}\left(\frac{(2k-10)k^{\frac{3}{2}}}{15}\right) > 0 \\ a_- &= \left| max\left\{F^{-1}\left(\frac{10k-2}{15}\right) < 0\right\}\right| \end{aligned}$$

If $k_\infty < k < k^\infty$ the system has no periodic solutions.

(iii) $\underline{0 \ll \epsilon \ll \infty}$: Numerical computations of this system suggest that the curves $k_*(\epsilon)$ and $k^*(\epsilon)$ behave as it is shown in Fig. 2. Thus we find that $k_\infty \le k_* \le k_0$ and $k^0 \le k^* \le k^\infty$ for every real ϵ.

In summary, if $k_0 < k < k^0$ there is no periodic solution, and, if $0 < k < k_\infty$ or $k > k^\infty$ the system has two limit cycles. Observe that $k^\infty < (7+\sqrt{45})/2$, and the requirement $k > (7+\sqrt{45})/2$ is not necessary for having two limit cycles for some ϵ, as it was proposed in [Lloyd, 1987].

Some amplitudes $a_\pm$ are given in the following table:

k	$\epsilon = 0$	$\epsilon = 1$	a
0	$\sqrt{2}$	1.41431	a_+
0	0	0	a_-
0.16	1.19001	1.19000	a_+
0.16	0.95072	0.95172	a_-
5	$\emptyset$	$\emptyset$	a_+
5	$\emptyset$	$\emptyset$	a_-
7	3.29065	3.29504	a_+
7	2.27410	2.44764	a_-

Example 3: $\underline{\mathbf{f}(\mathbf{x}) = \mathbf{5x^4} - \mathbf{3}\mu\mathbf{x^2} + \delta}$, where μ and δ are two real parameters (Fig. 3). This system is a generalization of Example 2. Several authors have studied the case $\delta = 1$: in Ref. [Rychkov, 1975], Rychkov shows that this equation have two cycles when $\epsilon > 0$ and $\mu > 2.5$; Alsholm has improved this result lowering the bound to $\mu > 2.3178$ ($\mu \geq 2.3178\ \delta^{\frac{1}{2}}$), and in [Odani, 1996], Odani obtained a sharper result $\mu > \sqrt{5}$. In [Giacomini & Neukirch, 1997], Giacomini & Neukirch obtain a sequence of algebraic approximations, in the parameter plane (δ, μ) for $\epsilon = 1$, to the bifurcation set $B_{\epsilon=1}(\delta, \mu) = 0$, where the system undergoes a saddle-node bifurcation. Here, the bifurcation curves $B_0(\delta, \mu) = 0$ and $B_\infty(\delta, \mu) = 0$ in the weakly and in the strongly nonlinear regimes are calculated, respectively. We obtain $B_0(\delta, \mu) = 9\mu^2 - 40\delta$ and $B_\infty(\delta, \mu) = \mu^2 - 5\delta$. For intermediate values of ϵ, numerical simulations show that the curves $B_\epsilon(\delta, \mu) = 0$ are localized between $B_0(\delta, \mu) = 0$ and $B_\infty(\delta, \mu) = 0$ in such a way that if $\epsilon_2 > \epsilon_1$ then $B_{\epsilon_2}(\delta, \mu) = 0$ is between $B_{\epsilon_1}(\delta, \mu) = 0$ and $B_\infty(\delta, \mu) = 0$. These results are in agreement with the earlier works cited above and let us a better understanding of the behaviour of this system for all the values of δ and μ.

(i) $\underline{\epsilon \longrightarrow 0}$: The amplitudes $a_\pm$ of the limit cycles in this regime are the positive solutions of the equation $\beta(a) = 0$, that is:

$$a_\pm = \left\{ \frac{1}{5} \left(3\mu \pm \sqrt{9\mu^2 - 40\delta} \right) \right\}^{\frac{1}{2}}$$

If $\delta < 0$ there is only one limit cycle of amplitude a_+. If $\delta > 0$ and $\mu < 0$ there are no real solutions for $a_\pm$. If $\delta > 0$ and $\mu > 0$, the sign of the discriminant $B_0(\delta, \mu) = 9\mu^2 - 40\delta$ determines the number of periodic solutions. If $3\mu < \sqrt{40}\ \delta^{\frac{1}{2}}$ there is none and if $3\mu > \sqrt{40}\ \delta^{\frac{1}{2}}$ the system has two limit cycles of amplitudes $a_\pm$. At $\delta = 0$ a limit cycle of small amplitude bifurcates from the origin after an Andronov-Hopf bifurcation. For the values (δ, μ) where $B_0(\delta, \mu) = 0$ the two limit cycles appear or disappear by a saddle-node bifurcation.

(ii) $\underline{\epsilon \longrightarrow \infty}$: The positive local maxima of $F(x) = x^5 - \mu x^3 + \delta x$ must be found. If $\delta < 0$ there is only a positive local maximum at $x_0 = -(\frac{3\mu+\Delta}{10})^{\frac{1}{2}}/\sqrt{10}$ where $\Delta = \sqrt{9\mu^2 - 20\delta}$. We have $F(x_0) = -(3\mu^2 + \mu\Delta - 20\delta)x_0/25$. The amplitude a_+ of this limit cycle is:

$$a_+ = F^{-1} \left[\left(\frac{3\mu^2 + \mu\Delta - 20\delta}{25} \right) \left(\frac{3\mu + \Delta}{10} \right)^{\frac{1}{2}} \right] > 0$$

If $\delta > 0$ and $3\mu < \sqrt{20}\ \delta^{\frac{1}{2}}$, $F(x)$ has no local maxima and the system has no periodic solutions. If $\delta > 0$ and $3\mu \geq \sqrt{20}\ \delta^{\frac{1}{2}}$ there are two positive local maxima localized at $x_+ = -(3\mu + \Delta)^{\frac{1}{2}}/\sqrt{10}$ and $x_- = (3\mu - \Delta)^{\frac{1}{2}}/\sqrt{10}$. The condition for having two limit cycles is: $F(x_+) > F(x_-)$. This follows if $\mu > \sqrt{5}\delta^{\frac{1}{2}}$. The amplitudes $a_\pm$ of the limit cycles are:

$$a_+ \quad = \quad F^{-1} \left[\left(\frac{3\mu^2 + \mu\Delta - 20\delta}{25} \right) \left(\frac{3\mu + \Delta}{10} \right)^{\frac{1}{2}} \right] > 0$$

$$a_{-} = \left| max\left\{ F^{-1}\left[\left(\frac{-3\mu^2+\mu\Delta+20\delta}{25} \right) \left(\frac{3\mu-\Delta}{10} \right)^{\frac{1}{2}} \right] < 0 \right\} \right|$$

If $\mu < \sqrt{5}\delta^{\frac{1}{2}}$ there are not periodic solutions. Then the bifurcation curve where the system undergoes the saddle-node bifurcation is defined by $B_\infty(\delta,\mu) = \mu^2 - 5\delta$.

(iii) $\underline{0 \ll \epsilon \ll \infty}$: Numerical simulations of this system suggest that the bifurcation curves $B_\epsilon(\delta,\mu) = 0$ are located between $B_0(\delta,\mu) = 0$ and $B_\infty(\delta,\mu) = 0$, in such a way that if $\epsilon_2 > \epsilon_1$ then $B_{\epsilon_2}(\delta,\mu) = 0$ is located between $B_{\epsilon_1}(\delta,\mu) = 0$ and $B_\infty(\delta,\mu) = 0$. If $\mu^*(\epsilon)$ is the solution of $B_\epsilon(\delta_0,\mu) = 0$ for a fixed δ_0, then $\sqrt{40\delta_0}/3 \le \mu^*(\epsilon) \le \sqrt{5\delta_0}$ and $\mu^*(\epsilon_2) > \mu^*(\epsilon_1)$ if $\epsilon_2 > \epsilon_1$.

As an example we give some values of the amplitudes $a_\pm$ of the periodic solutions for $\mu = 1$:

δ	$\epsilon = 0$	$\epsilon = 1$	a
-1	$\sqrt{2}$	1.40990	a_+
0.1	1.02334	1.02344	a_+
0.1	0.39087	0.39090	a_-
0.3	$\emptyset$	$\emptyset$	a_+
0.3	$\emptyset$	$\emptyset$	a_-

Example 4: $\underline{\mathbf{f(x) = 7x^6 - 5(29+b^2)x^4 + 3(100+29b^2)x^2 - 100b^2}}$, where b is a real parameter (Fig. 4). Giacomini & Neukirch have investigated this system in Ref. [Giacomini & Neukirch, 1997]. They find that the solutions of the equation $F(x,b) = 0$ do not give the right qualitative amplitude-bifurcation diagram, where $F(x,b) = x(x^2-b^2)(x^2-2^2)(x^2-5^2)$. In fact, the plot of the roots of $F(x,b) = 0$ announce the presence

of a transtricital bifurcation near $b = 2$ and $b = 5$, and an Andronov-Hopf bifurcation at $b = 0$. Their method of algebraic approximations to the bifurcation curves shows that the supposed transcritical bifurcations are indeed saddle-node bifurcations. They conclude that the system can have one or three limit cycles. We confirm these results and stablish the correct amplitude-bifurcation curves $a_i^\epsilon(b)$, $i = 1, 2, 3$, for the three limit cycles, in the weakly, $a_i^0(b)$, and in the strongly, $a_i^\infty(b)$, nonlinear regimes. The values of b for which the saddle-node bifurcations occur are calculated in those regimes. Numerical simulations show that for intermediate values of ϵ the amplitude-bifurcation $a_i^\epsilon(b)$ curves and the values of b for which the saddle-node bifurcations occur are localized in the regions bounded by the curves $a_i^0(b)$ and $a_i^\infty(b)$, $i = 1, 2, 3$. As in the former examples, the variation of parameter ϵ does not introduce new qualitative information in the system, and only produces slight perturbations in the amplitude-bifurcation diagrams.

(i) $\underline{\epsilon \longrightarrow 0}$: Mapple calculations allow us to solve the equation $\beta(a) = 0$ for the amplitudes $a_i^0(b)$, $i = 1, 2, 3$. The number of positive real solutions of that cubic equation in a^2 is determined by the the sign of the polynomial $\Delta(b) = -0.01784\ b^8 + 1.15301\ b^6 - 21.65794\ b^4 + 132.559\ b^2 - 189.45$. If $\Delta < 0$ there are three limit cycles and if $\Delta > 0$ there is only one. If $\Delta = 0$ a saddle-node bifurcation arises in the system. The positive roots of $\Delta(b)$ are: $b_1^0 = 1.42636$, $b_2^0 = 2.84148$, $b_3^0 = 4.17545$ and $b_4^0 = 6.08945$. If $0 < b < b_1^0$, $b_2^0 < b < b_3^0$ or $b > b_4^0$ then $\Delta(b) < 0$ and there are three periodic solutions . If $b_1^0 < b < b_2^0$ or $b_3^0 < b < b_4^0$ then $\Delta(b) > 0$ and there is only one. The amplitudes of the limit cycles for a given $\bar{b}$ are the cuts

of the line $b = \bar{b}$ with the curves $a_1^0(b)$, $a_2^0(b)$ and $a_3^0(b)$ (see Fig. 4).

(ii) $\underline{\epsilon \to \infty}$: Numerical calculation of show that the values of b for which the system undergoes a saddle-node bifurcation are: $b_1^\infty = 1.21$, $b_2^\infty = 3.49$, $b_3^\infty = 3.95$ and $b_4^\infty = 6.40$. If $0 < b < b_1^\infty$, $b_2^\infty < b < b_3^\infty$ or $b > b_4^\infty$ the system has three limit cycles, and if $b_1^\infty < b < b_2^\infty$ or $b_3^\infty < b < b_4^\infty$ there is only one periodic solution (see Fig. 4).

(iii) $\underline{0 \ll \epsilon \ll \infty}$: Numerical computations show that the amplitude-curves $a_i^\epsilon(b)$, $i = 1, 2, 3$, are localized in the narrow shaded region bounded by $a_i^0(b)$ and $a_i^\infty(b)$ (Fig. 4a). Remark that the behaviour of the system suggests, once more, that the number of limit cycles do not increase when ϵ increases (Fig. 4b).

4 Conclusions

Limit cycles are isolated periodic solutions of specific nonlinear differential equations and can model self-sustained oscillations in nature. There are two difficult and connected problems in relation with limit cycles: the determination of bifurcation curves of these solutions in the parameter space and the determination of the maximal number of such solutions. In this work, we have exploited the possibility of calculating the bifurcation curves of the limit cycles of Liénard equation $\ddot{x} + \epsilon f(x)\dot{x} + x = 0$ in the weakly ($\epsilon \to 0$) and in the strongly ($\epsilon \to \infty$) nonlinear regimes when the viscous term $f(x)$ is even. Firstly, these calculations allow us to improve the results existing in the literature for different examples in

these regimes. Secondly, the systems analyzed seem to follow the same pattern: the number of limit cycles does not increase when the nonlinearity ϵ increases. Moreover, the bifurcation curves for intermediate ($0 \ll \epsilon \ll \infty$) nonlinearity are always located between the bifurcation curves corresponding to the two extreme regimes. This means that although the variation of the nonlinearity ϵ introduces an important modification of the time scale and wave form of the oscillation, it perturbs slightly its amplitude. Only if two limit cycles have a very close amplitude for a given ϵ, there exists the possibility of collapse of those limit cycles by a saddle-node bifurcation when $|\epsilon|$ increases. If the system loses these two limit cycles it do not recover them for a stronger nonlinearity ϵ. If the amplitudes of the limit cycles are separated enough for a given ϵ, the number of periodic motions is conserved when ϵ is varied.
In particular, if we restrict ourselves to even-polynomial viscous forces, this behaviour suggests that Lins-Melo-Pugh conjecture on the number of limit cycles of Liénard systems is true. This is so because the conjecture is true in the weakly nonlinear regime and, according to the behaviour above explained, it should be true for any other regime.

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Figure Captions

Figure 1: A typical phase portrait of Eq. (3). The limit cycles of amplitudes a_i, $i = 1, 2, \cdots$, enclose the origin and have the symmetry $(x, y) \leftrightarrow (-x, -y)$. Stable and unstable limit cycles alternate.

Figure 2: Qualitative bifurcation diagram in the parameter plane (k, ϵ) of system (2) for $f(x)$ given in Example 2. In region I, where $k < 0$, there is only one periodic solution; in region II, where $0 < k < k_*(\epsilon)$ or $k > k^*(\epsilon)$, there are two limit cycles, and in region $\bigcirc$, where $k_*(\epsilon) < k < k^*(\epsilon)$, there are none. On the line $k = 0$ the system undergoes an Andronov-Hopf bifurcation and on the curves $k_*(\epsilon)$ and $k^*(\epsilon)$ a saddle-node bifurcation. (Nomenclature in the text: $k_*(0) \equiv k_0$, $k^*(0) \equiv k^0$, $k_*(\infty) \equiv k_\infty$ and $k^*(\infty) \equiv k^\infty$).

Figure 3: The complete bifurcation diagram of system (2) for $f(x)$ given in Example 3. The system has no periodic solutions in region $\bigcirc$, one limit cycle in region I and two limit cycles in region II. On the line $\delta = 0$ the system undergoes an Andronov-Hopf bifurcation, and on the curves $B_0(\delta, \mu) = 9\mu^2 - 40\delta = 0$ and $B_\infty(\delta, \mu) = \mu^2 - 5\delta = 0$ a saddle-node bifurcation arises for $\epsilon = 0$ and $\epsilon = \infty$, respectively. In region B are located all the bifurcation curves $B_\epsilon(\delta, \mu)$: the curve B_{ϵ_2} is located between B_{ϵ_1} and B_∞ if $\epsilon_1 < \epsilon_2$.

Figure 4: Bifurcation curves of system (2) for $f(x)$ given in Example 4: **(a)** Amplitude bifurcation-diagram where $a_i^0(b)$ correspond to $\epsilon = 0$ and $a_i^\infty(b)$ to $\epsilon = \infty$, $i = 1, 2, 3$. The amplitude-bifurcation $a_i^\epsilon(b)$ curves are located in the interior of the shaded region for every ϵ. The number and amplitudes of limit cycles for $b = \bar{b}$ are the number and a-coordinates of the intersections between the curves $a_i^\epsilon(b)$ and the line $b = \bar{b}$. (solid lines correspond to stable cycles and dashed lines to unstable ones for $\epsilon > 0$).

(b) Qualitative bifurcation diagram in the parameter plane (b, ϵ). In region I there is only a periodic solution and in region III there are three limit cycles.